Eigen DLA Read Me – This is a Forward Stretch DLA

Who: This DLA is designed for students who have completed the Systems of REs lessons, but not the Eigenvalue and Eigenvector Lessons of MA103.

Emphasis: This DLA has an emphasis on matrix and vector computations and understanding eigenvalues and vectors.

Stretch: This DLA asks students to discover and define eigenvalues and eigenvectors without having been introduced to them in class.

Timing: Individual – 15 Minutes; Team - 31 Minutes; Reflection - 7 Minutes

Technology: This is a NO tech examination! They can use their issued calculators. The intent of this choice is to ensure that students understand the mathematics behind eigenvalues and eigenvectors by discovering and using the relationship $A\vec{v} = \lambda\vec{v}$ along with graphical interpretations.

Grading: Focus is placed on matrix and vector computations and properties.

General Overview:

- As a read ahead, students will read about the number of computations required to iterate a system very far. They will see Mathematica break trying to perform these computations and will review the analytic solution of a single recursion equation as they think about how to reduce the number of necessary computations.
- As individuals, students will perform several matrix and vector computations and give the system of recursion equations that goes with a given state diagram.
- As teams, students will count computations for taking matrices to powers, explore a matrix times a vector when the vector is an eigenvector and when it is not. They will use these examples to make observations about eigenvectors and eigenvalues, ultimately using those observations to develop their own definitions of eigenvectors and eigenvalues.
- As individuals, students will synthesize their knowledge by taking a given matrix and vector and determining if the vector is an eigenvector and finding its associated eigenvalue.

When: This DLA is most appropriate in the second half of Block II of MA 103.

Component	Problem	Possible Points	Topic
Individual	1	16	Vector and Matrix Multiplication
	2	4	System of Recursion Equations
Team	3	6	Counting Computations
	4	3	Matrix×Vector Computation & Vector Graphing
	5	3	
	6	3	
	7	4	Eigenvector and Eigenvalue Discovery
	8	6	Eigenvector and Eigenvalue Definition
Reflection	9	5	Eigenvector and Eigenvalue Recognition
	Total	50	

The following is correct code for a System of Recursion Equations in Mathematica. However, when it is evaluated, Mathematica returns an error.

```
p[n_] := T.p[n - 1] + d;

T = ( 0.9  0.5  0    0  0
      0    0.4  0.9  0  0
      0.1  0    0    0  0
      0    0    0    1  0
      0    0.1  0.1  0  1 );

d = ( -51 000
      0
      12 500
      38 500
      0 );

p[0] = ( 25 000 000
         0
         0
         0
         0 );

p[1040]

··· $RecursionLimit: Recursion depth of 1024 exceeded during evaluation of T.p(18 - 1).

Hold[d + T.p(1040 - 1)]
```

This error means that we are asking Mathematica to perform computations beyond its capabilities. We will explore why this is beyond Mathematica's capabilities and how we can work around it.

Even Mathematica can't deal with computations when the number of computations gets too large. In the example provided, Mathematica was being asked to perform 286,000 computations. We need a better way to iterate systems of recursion equations.

When we needed a better way to iterate a single recursion equation, we used an analytic solution. Review analytic solutions of a single recursion equation in Section 1.9 of MRCW. Think about how this might apply to a system of recursion equations and how it might reduce the number of computations necessary to compute $\overrightarrow{p_{1040}}$.

Expectations for preparation: Techniques include looking at lesson and block objectives, reviewing course material in this block, make connections to the course material covered thus far, do additional problems that reinforce the connection made to the critical concepts in the block, and lastly make a sheet of notes as needed.

INDIVIDUAL

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Section: _____________

DLA 2B Eigen – MA103 50 Points

GENERAL INSTRUCTIONS: Read all instructions carefully.

1. You have 55 minutes to complete the DLA. You will have 15 minutes for the first individual portion, 31 minutes for the team portion, and 7 minutes for the individual reflection portion.
2. Early departure from the individual reflection portion is authorized. Give the DLA to your instructor or place it on your instructor's desk when completed.
3. Authorized items: One sheet (normal 8"x11" paper, front and back) of hand-written notes and your issued calculator.
4. Items not authorized: your computer, any type of phone, or other electronic device.
5. Including this cover sheet, there are 9 pages to the DLA.
6. Clearly indicate your answer by underlining or boxing your solution (e.g. $\underline{0 < x < 5}$, or $\boxed{0 < x < 5}$).
7. **Show your work**. Partial credit can only be awarded if you **show your work**. It is always best to show intermediate steps to illustrate your problem-solving process (i.e. $p_1 = ;\ p_2 =$).
8. Use a blank continuation sheet if you need more space and clearly identify that the problem is continued both on the DLA and on the continuation sheet. Be sure to place your name on all continuation sheets.

Component	Problem	Possible Points	Points Earned
Individual	1	16	
	2	4	
Team	3	6	
	4	3	
	5	3	
	6	3	
	7	4	
	8	6	
Reflection	9	5	
	Total	50	

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DLA 2B Eigen – MA103 50 Points

Show your work

1. [16 Points] Given the following matrix and vectors, perform each of the following computations.

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \vec{w} = \begin{bmatrix} 1 & -1 \end{bmatrix}$$

a. $3\vec{v} =$

b. $\vec{w} \cdot \vec{v} =$

c. $A\vec{v} =$

d. $AA =$

e. $A^2 =$

f. $A^3 =$

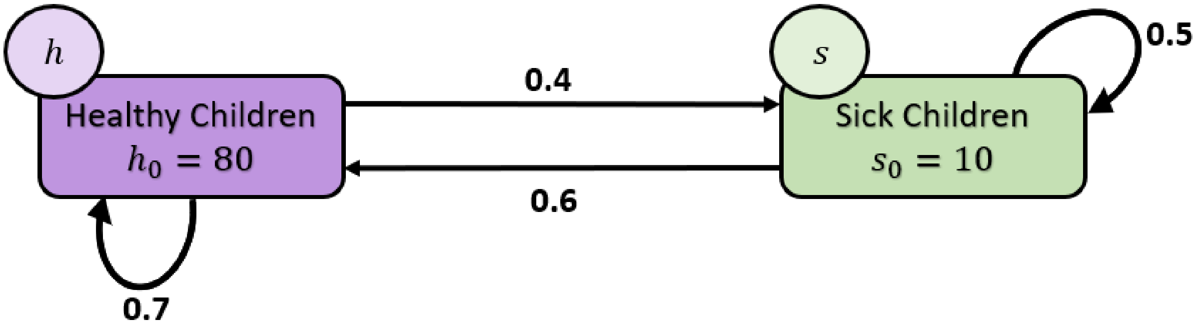
Show your work

2. [4 Points] Write the system of recursion equations that goes with the following state diagram and definition of variables.

Variables:

h_n is the number of healthy children in Arneyville after n months.
s_n is the number of sick children in Arneyville after n months.

State Diagram:



Show your work

In the Read Ahead, you revisited the analytic solution for a single recursion equation and considered how we could do something similar with a system of recursion equations. It would be great if we could use $\overrightarrow{p_n} = T^n(\overrightarrow{p_0} - \overrightarrow{p_*}) + \overrightarrow{p_*}$. Let's explore the T^n part of that.

3. [6 Points] When taking the dot product of $\vec{v} = \langle v_1, v_2 \rangle$ and $\vec{u} = \langle u_1, u_2 \rangle$, the arithmetic required is two multiplications and one addition. Further, when taking the dot product of $\vec{r} = \langle r_1, r_2, r_3 \rangle$ and $\vec{s} = \langle s_1, s_2, s_3 \rangle$, the arithmetic required is three multiplications and two additions. Using this information, complete the following table referencing matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and matrix $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$.

Operation	Number of Dot Products	Number of Multiplications	Number of Additions	Total Number of Computations
A^2				
A^3				
B^2				
B^3				

When T is a 5×5 matrix and we are computing $\overrightarrow{p_{1040}}$, it turns out that this approach only reduces the workload for Mathematica from 286,000 computations to 234,055 computations. We need a better approach. It would be ideal if we could replace a matrix with a single scalar value. That would allow us to reduce the number of computations required for the T^{1040} part from 234,000 to 1,039 computations. Let's explore this possibility.

TEAM

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DLA 2B Eigen – MA103 50 Points

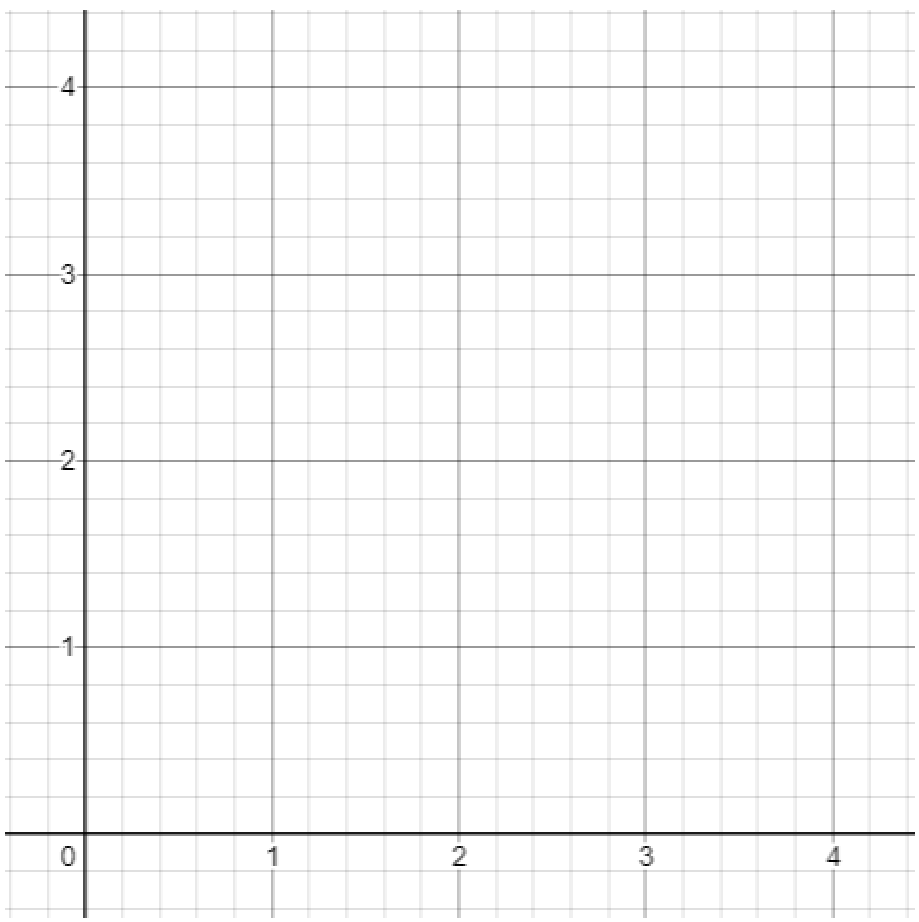
Show your work

In the following problems, use the matrix $M = \begin{bmatrix} 0.5 & 0.4 \\ 0.6 & 0.7 \end{bmatrix}$.

4. [3 Points] For $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

a. Compute $M\vec{v}$.

b. Plot $\vec{v}$ and $M\vec{v}$ on the axes provided below.



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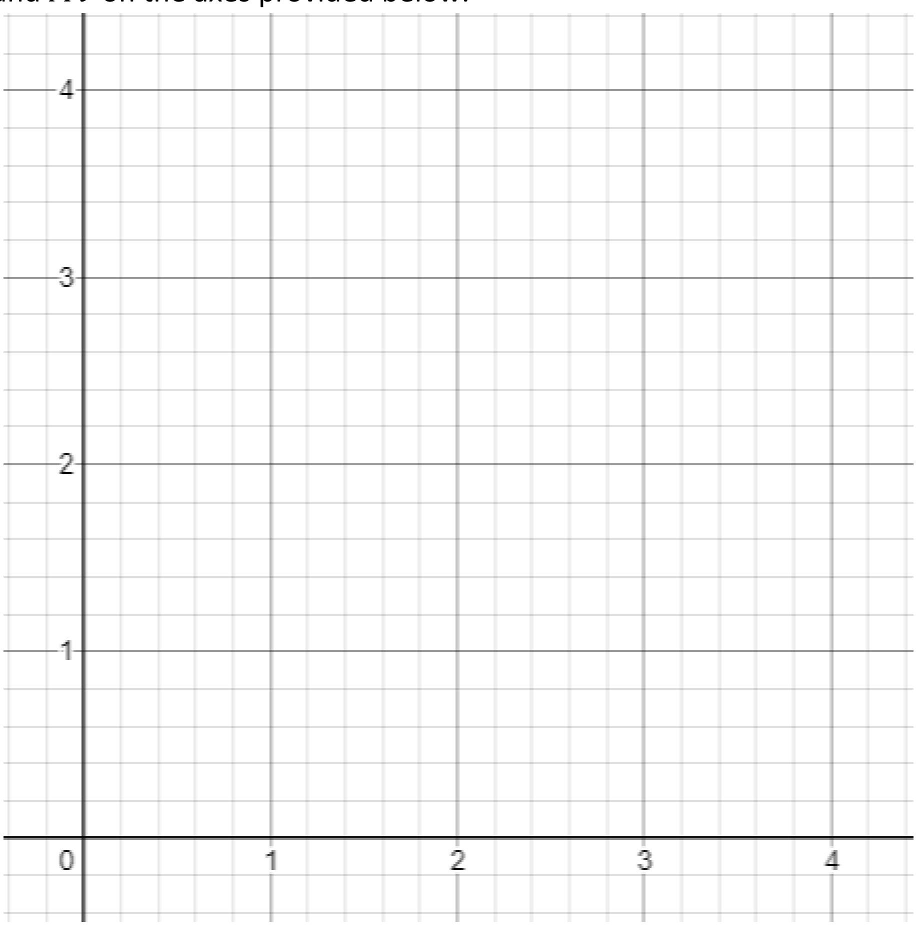
DLA 2B Eigen – MA103 50 Points

Show your work

5. [3 Points] For $\vec{v} = \begin{bmatrix} 1.5 \\ 2 \end{bmatrix}$

a. Compute $M\vec{v}$.

b. Plot $\vec{v}$ and $M\vec{v}$ on the axes provided below.



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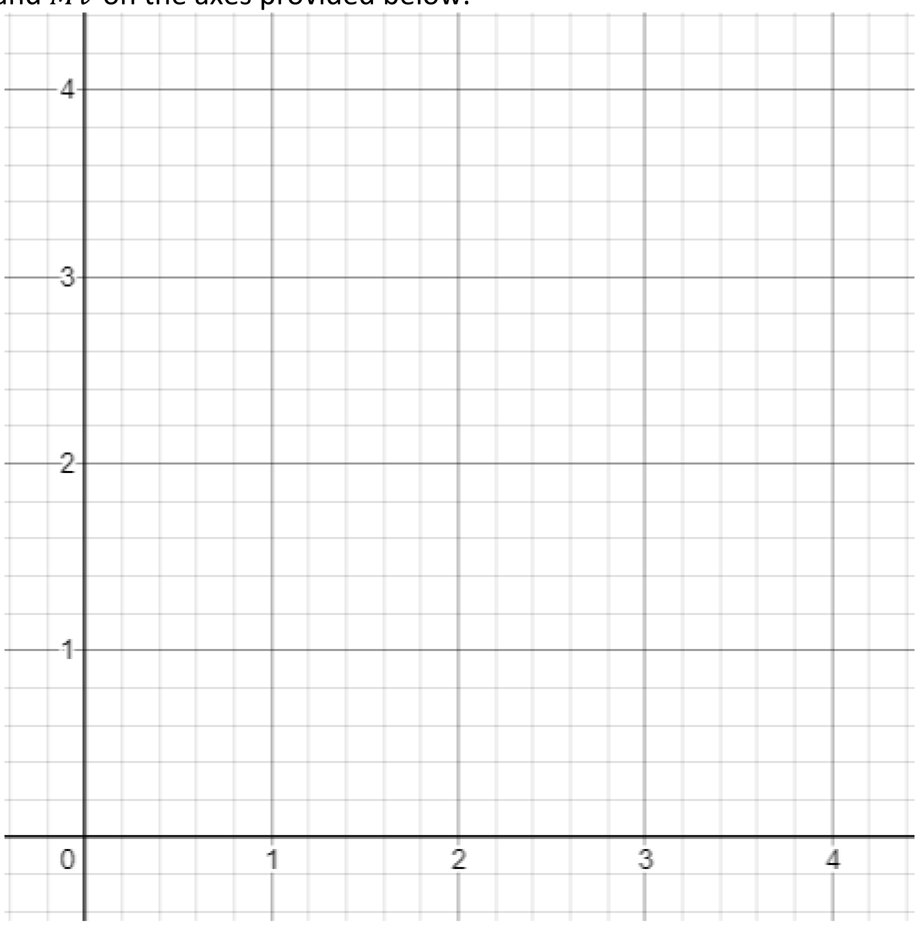
DLA 2B Eigen – MA103 50 Points

Show your work

6. [3 Points] For $\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

a. Compute $M\vec{v}$.

b. Plot $\vec{v}$ and $M\vec{v}$ on the axes provided below.



TEAM

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DLA 2B Eigen – MA103 50 Points

Show your work

7. [4 Points] Consider your responses to numbers 4-6.
   a. What is different about the plot in number 6 compared to numbers 4 and 5?

   b. What is different about the relationship between $\vec{v}$ and $M\vec{v}$ in your computation for number 6 compared to your computations for numbers 4 and 5?

8. [6 Points] The scalar that you found in number 7b is called an Eigenvalue and the vector $\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is called an Eigenvector.
   a. Use what you have discovered to define Eigenvalue.

   b. Use what you have discovered to define Eigenvector.

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DLA 2B Eigen – MA103 50 Points

Show your work

9. [5 Points] Is $\vec{w} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ an Eigenvector of the matrix $M = \begin{bmatrix} 0.5 & 0.4 \\ 0.6 & 0.7 \end{bmatrix}$? If so, how do you know and what is its associated Eigenvalue? If not, why not?